HYDRODYNAMIC ANALYSIS OF THE HOT ROLLING OF SOME POLYMER MATERIALS

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The general flow-line picture is examined. Production of sheet with thickness between set limits is discussed. Calculation formulas are given.

Here I present a reasonably complete theoretical study of the rolling of a material that obeys

$$\tau = \mu_0 |\dot{\gamma}|^{m-1}\,\dot{\gamma}.$$

This power law has been chosen for simplicity. Rheological studies on polymers show that many of them are well described by this formula over a wide range, especially various compositions based on PVC and polyethylene [1], as well as filled rubber mixtures [2].

There are fairly many papers on calandering; the material has [3-5] been considered as a Newtonian fluid, or as a non-Newtonian one [6, 7] that obeys a power law, but with a constant effective viscosity, so that the problem reduces to the flow of a Newtonian fluid. Other papers [8-10] deal with the flow of Newtonian liquids having various types of relation between τ and $\dot{\gamma}$.

The following assumptions were made in deriving the basic equations: 1) the problem is two-dimensional; 2) the fluid is incompressible; 3) the flow is laminar; 4) the motion is of steady-state type; 5) the inertial forces are small relative to the viscous ones; 6) the gravitational forces are small relative to the viscous ones; 7) if the velocity component u(x, y) along the Ox axis is proportional to U, then v (the velocity component along the Oy axis) is proportional to Ul/L, where L and l are the characteristic lengths along the Ox and Oy axes, respectively, with $L \gg l$; 8) $\partial u/\partial x \propto U/L$; 9) $\partial u/\partial y \propto U/l$; 10) $\partial v/\partial x \propto Ul/L^2$. The first six assumptions give the basic equations [11] as:

$$0 = -\frac{\partial p}{\partial x} + \mu_0 M^{m-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mu_0 (m-1) M^{m-2}\left[2\frac{\partial u}{\partial x}\,\frac{\partial M}{\partial x} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial M}{\partial y}\right],$$

$$0 = -\frac{\partial p}{\partial y} + \mu_0 M^{m-1}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \mu_0 (m-1) M^{m-2}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial M}{\partial x} + 2\frac{\partial v}{\partial y}\,\frac{\partial M}{\partial y}\right],$$

where

$$M = \sqrt{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]}.$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

The components of the stress tensor are

$$\sigma_{xx} = -p + 2\mu_0 M^{m-1}\frac{\partial u}{\partial x},\quad \sigma_{xy} = \sigma_{yx} = \tau = \mu_0 M^{m-1}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\quad \sigma_{yy} = -p + 2\mu_0 M^{m-1}\frac{\partial v}{\partial y}.$$

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Conditions 8)-10) may be used with the usual estimates of boundary-layer theory to give the latter relations the form

$$p = p(x), \tag{1}$$

$$\frac{dp}{dx} = m\mu_0 \left| \frac{\partial u}{\partial y} \right|^{m-1} \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$\tau = \mu_0 \left| \frac{\partial u}{\partial y} \right|^{m-1} \frac{\partial u}{\partial y}. \tag{4}$$

These equations may be derived from those given in [9].

Figure 1 shows how the rolls operate. We have to find the pressure and velocity distributions that fit (1)-(3) in the region $-x_1 \le x \le x_s$ and $-h \le y \le +h$, where $x_s > 0$ is the roll insertion depth at which calandering starts, which we assume as given. Also, x_1 will be determined in the subsequent investigation. Figure 1 shows that $h = h_0 + R - \sqrt{R^2 - x^2}$. The rolls rotate at a constant angular velocity w (assumption 11), and we assume that the material adheres to the rolls (assumption 12). Then

$$\text{for } y = 0,\ v = 0\ \ \frac{\partial u}{\partial y} = 0 \text{ (flow symmetry)}, \tag{5}$$

$$\text{for } y = h,\ u = -\omega(R + h_0 - h)\ \ v = -\omega x \text{ (adhesion condition)},$$

$$\text{for } x = -x_1\ \ p = 0 \text{ (assumption)}, \tag{6}$$

$$x = x_s\ \ p = 0 \text{ (assumption)}.$$

There are no negative pressures in the relevant region (assumption 15).

It has been assumed [12] in discussing lubrication theory that, if the lubricant layer breaks with $dp/dx \ne 0$, then the break occurs in an unstable position and the break point moves in the direction of decreasing pressure. Assumption 16 is that $dp/dx = 0$ at the point where the material breaks away.

We give briefly the results of [13]. We integrate (2) and use (5) and (6) to get

$$u = \frac{m}{m+1}\left(\frac{1}{\mu_0}\right)^{\frac{1}{m}} \left|\frac{dp}{dx}\right|^{\frac{1}{m}} \left(y^{\frac{m+1}{m}} - h^{\frac{m+1}{m}}\right) - \omega(R + h_0 - h). \tag{7}$$

The flow rate at any point is

$$Q = -2\int_0^h u\,dy = \frac{2m}{2m+1}\left(\frac{1}{\mu_0}\right)^{\frac{1}{m}} \left|\frac{dp}{dx}\right|^{\frac{1}{m}}_{x=-x_1} h^{\frac{2m+1}{m}} + \omega(R + h_0 - h)\,h.$$

We equate the flow rates in any section and in the section where the calandering stops ($x = -x_1$) (assumption 2), which gives a differential equation for p:

$$\frac{m}{2m+1}\ \frac{1}{\mu_0}\left(\frac{dp}{dx}\right)^{\frac{1}{m}} h^{\frac{2m+1}{m}} = \frac{m}{2m+1}\left(\frac{1}{\mu_0}\right)^{\frac{1}{m}} \left|\frac{dp}{dx}\right|^{\frac{1}{m}}_{x=-x_1} h_1^{\frac{2m+1}{m}} + \omega(h_1 - h)(R + h_0 - h_1 - h).$$

Assumptions (13) and (16) give the following equations for dp/dx and $p(x)$:

$$\frac{dp}{dx} = A\,\frac{(h_1 - h)\,|h_1 - h|^{m-1}\,(h_2 - h)\,|h_2 - h|^{m-1}}{h^{2m+1}}, \tag{8}$$

$$p(x) = A\int_{-x_1}^{x} \frac{(h_1 - h)\,|h_1 - h|^{m-1}\,(h_2 - h)\,|h_2 - h|^{m-1}}{h^{2m+1}}\,dx, \tag{9}$$

where $A = \mu_0[(2m+1)/m]^m\,\omega^m$; $h_2 = R + h_0 - h_1$; $h_2 > h_1$.

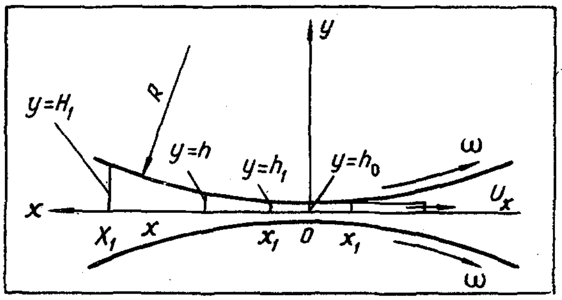

Fig. 1

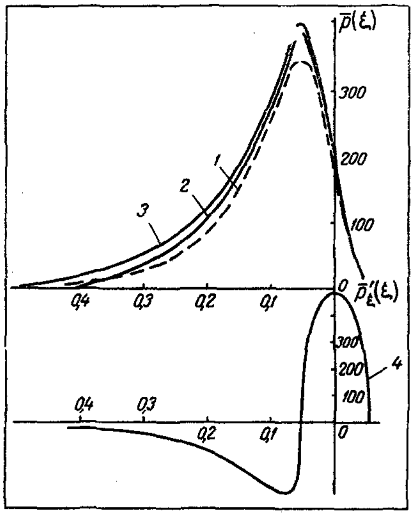

Fig. 2

Fig. 1. Scheme of roll operation.

Fig. 2. Graphs of $\bar{p}(\xi)$ and $dp/d\xi$: 1) k = 1.22, m = 1.0; 2) 1.22 and 0.31; 3) 1.24 and 0.31; 4) 0.22 and 0.31. $\eta_0 = 0.00625$.

We see from (8) that p(x) touches the Ox axis at $x = -x_1$, has a maximum at $x = +x_1$, and a minimum at $x = x_2 (h = h_2)$. Assumption 14) gives the following relation for h_1 (or x_1):

$$\int_{-x_1}^{x_s} \frac{(h_1 - h)\,|h_1 - h|^{m-1}\,(h_2 - h)\,|h_2 - h|^{m-1}}{h^{2m+1}}\,dx = 0. \tag{10}$$

Consider the behavior of p(x) as a function of h_1. We assume that there is a value $h_1 = h_{1l}$ $(x = x_{1l})$ such that p(x) touches the Ox axis at $x = x_{2l}$, $x_{2l} > x_{1l}$. Then h_{1l} and h_{2l} $(x = x_{2l})$ are defined by

$$\frac{dp}{dx} = 0,\ p = 0 \quad \text{or} \quad h_{2l} = R + h_0 - h_{1l}$$

and

$$\int_{-x_{1l}}^{x_{2l}} \frac{(h_{1l} - h)\,|h_{1l} - h|^{m-1}\,(h_{2l} - h)\,|h_{2l} - h|^{m-1}}{h^{2m+1}}\,dx = 0.$$

Let x_s satisfy $x_{2l} < x_s < x_{1l}$; then there will be a value h_1 that satisfies assumptions 13), 14), and 15). If $h_1 = h_0$, we see that p(x) will be negative for x > 0. Also, h_1 varies within the limits $h_{1l} > h_1 > h_0$. We introduce the following dimensionless quantities: $h/R = \eta$, $x/R = \xi$, $x_1/R = \xi_1$, $h_1/h_0 = k$; then (8) and (9) may be put as follows (Fig. 2):

$$\frac{dp}{dx} = \frac{A}{R}\,\frac{d\bar{p}}{d\xi},$$

where

$$\frac{d\bar{p}}{d\xi} = \{[\eta_0(k-1) + \sqrt{1-\xi^2} - 1]\,|\eta_0(k-1) + \sqrt{1-\xi^2} - 1|^{m-1}$$
$$\times(\sqrt{1-\xi^2} - k\eta_0)\,|\sqrt{1-\xi^2} - k\eta_0|^{m-1}\}\,\{(1+\eta_0 - \sqrt{1-\xi^2})^{2m+1}\}^{-1}; \tag{11}$$
$$p = A\bar{p}(\xi),$$

and

$$\bar{p}(\xi) = \int_{-\xi_1}^{\xi_s} \{[\eta_0(k-1) + \sqrt{1-\xi^2} - 1]\,|\eta_0(k-1) + \sqrt{1-\xi^2} - 1|^{m-1}$$
$$\times(\sqrt{1-\xi^2} - k\eta_0)\,|\sqrt{1-\xi^2} - k\eta_0|^{m-1}\}\,\{(1+\eta_0 - \sqrt{1-\xi^2})^{2m+1}\}^{-1}. \tag{12}$$

If we replace integration along the arc by the approximation of integration along a straight line, the reaction on the rolls is

$$P = L \int_{-x_1}^{x_s} p dx = LR\overline{P},$$

where

$$\overline{P} = \int_{-\xi_1}^{\xi_s} \overline{p}(\xi)\, d\xi;$$

and L is the working width of the rolls. Then (2)-(4) give the stress at y = h due to viscous friction as $\tau = (dp/dx)h$ or $\tau = A\overline{\tau}$, where $\overline{\tau} = (d\overline{p}/d\xi)\eta$. The frictional force per unit roll length is

$$F = \int_{-x_1}^{x_s} h \left| \frac{dp}{dx} \right| dx = AR\overline{F},$$

where

$$\overline{F} = \int_{-\xi_1}^{\xi_s} h \left| \frac{dp}{d\xi} \right| d\xi,$$

and M = FRL is the torque due to the viscous friction.

Consider the flow-line pattern. We substitute (8) into (7) to get

$$u = \frac{2m+1}{m+1} \omega \frac{1}{h} \left\{ [h^2 - h(R + h_0) + h_1 h_2] \left[\left(\frac{y}{h} \right)^{\frac{m+1}{m}} - 1 \right] - \frac{m+1}{2m+1} hh_2 \right\}. \tag{13}$$

This gives $\partial u/\partial x$. It follows from (3) that this expression equals $-\partial v/\partial y$. We integrate the expression for $\partial v/\partial y$ with respect to y and use (6) to get

$$v = \frac{\omega}{m+1} \frac{1}{h} \frac{dh}{dx} \frac{y}{h}$$

$$\times \left\{ [h^2 - (m+1)(R + h_0) h - (2m+1) h_1 h_2] \left[\left(\frac{y}{h} \right)^{\frac{m+1}{m}} - 1 \right] - (m+1) h (R + h_0 - h) \right\}. \tag{14}$$

The following is the differential equation for the flow lines:

$$\frac{dy}{dx} = \left[\frac{y}{h} \left\{ [h^2 - (m+1)(R + h_0) h + (2m+1) h_1 h_2] \left[\left(\frac{y}{h} \right)^{\frac{m+1}{m}} - 1 \right] \right.\right.$$

$$\left.\left. - (m+1) h (R + h_0 - h) \right\} \right] \left[(2m+1) [h^2 - h(R + h_0) + h_1 h_2] \right.$$

$$\left. \times \left[\left(\frac{y}{h} \right)^{\frac{m+1}{m}} - 1 \right] - (m+1) h (R + h_0 - h) \right]^{-1} \frac{dh}{dx}. \tag{15}$$

The following equations give the coordinates of the singularity $M(x_M, y_M)$:

$$[h^2 - (m+1)(R + h_0) + (2m+1) h_1 h_2] \left[\left(\frac{y}{h} \right)^{\frac{m+1}{m}} - 1 \right] - (m+1) h (R + h_0 - h) = 0,$$

$$(2m+1) [h^2 - h(R + h_0) + h_1 h_2] \left[\left(\frac{y}{h} \right)^{\frac{m+1}{m}} - 1 \right] - (m+1) h (R + h_0 - h) = 0.$$

Then

$$h_M = \frac{R + h_0}{2},$$

$$y_M = \frac{R+h_0}{2}\left\{\frac{m(R+h_0)^2-4(2m+1)h_1h_2}{(2m+1)[(R+h_0)^2-4h_1h_2]}\right\}^{\frac{m}{m+1}}$$

and from

$$h = R + h_0 - \sqrt{R^2 - x_M^2},$$

$$y=0,\ (2m+1)[h^2 - h(R+h_0) + h_1h_2] + (m+1)h(R+h_0-h) = 0,$$

$$mh^2 - m(R+h_0)h + (2m+1)h_1h_2 = 0,$$

whence

$$H_{1,2} = \frac{R+h_0}{2} \mp \frac{1}{2}\sqrt{(R+h_0)^2 - 4\frac{2m+1}{m}h_1h_2}.$$

We assume that

$$(R+h_0)^2 - 4\frac{2m+1}{m}h_1h_2 \geqslant 0. \tag{16}$$

We see from (13) and (14) that we have branch points at $N_1(X_1, 0)$ and $N_2(X_2, 0)$, where X_1 and X_2 are values of x corresponding to H_1 and H_2. We now show that point M is a center, for which purpose we make a parallel transfer of the coordinate axes, placing the origin at M, the conversion formulas being $\bar{x} = x + x_M$ and $\bar{y} = y + y_M$. The functions in the numerator and denominator on the right in (15) are continuous, as are their partial derivatives with respect to x and y within the relevant region. Equation (15) can be put in the following form [14]:

$$\frac{d\bar{y}}{d\bar{x}} = \frac{a\bar{x} + b\bar{y} + \varphi(\bar{x},\bar{y})}{c\bar{x} + d\bar{y} + \psi(\bar{x},\bar{y})},\ \lim_{\substack{\bar{x}\to 0\\ \bar{y}\to 0}} \frac{\varphi(\bar{x},\bar{y})}{\sqrt{\bar{x}^2+\bar{y}^2}} = 0,\ \lim_{\substack{\bar{x}\to 0\\ \bar{y}\to 0}} \frac{\psi(\bar{x},\bar{y})}{\sqrt{\bar{x}^2+\bar{y}^2}} = 0,$$

$$a = \frac{1}{4}(m+1)\left(\frac{R+h_0}{2}\right)^{\frac{1}{m}} y_M$$

$$\times\left\{\frac{2m}{2m+1}\,\frac{R+h_0}{(R+h_0)^2-4h_1h_2} + \frac{1}{m}[m(R+h_0)^2 - 4(2m+1)h_2]\right\}\frac{(R-h_0)(3R+h_0)}{R+h_0},$$

$$c = -b = \frac{m+1}{4m}\left(\frac{R+h_0}{2}\right)^{\frac{m+1}{m}}[m(R+h_0)^2 - 4(2m+1)h_1h_2]\sqrt{\frac{(R-h_0)(3R+h_0)}{R+h_0}},$$

$$d = -\frac{(2m+1)(m+1)}{4}\left(\frac{R+h_0}{2}\right) y_M^{\frac{1}{m}}[(R+h_0)^2 - 4h_1h_2].$$

From (16) we conclude that $y_M > 0$, $a > 0$, $b < 0$, $c > 0$, $d < 0$. We have:

$$c + b = 0,$$

$$(c-b)^2 + 4da = -\frac{1}{2}(m+1)^2\left(\frac{R+h_0}{2}\right)^{\frac{2(m+1)}{m}}$$

$$\times (R+h_0)^3(R-h_1)(3R+h_0)\frac{m(R+h_0)^2 - 4(2m+1)h_1h_2}{(2m+1)[(R+h_0)^2 - 4h_1h_2]} < 0.$$

We would have a center at M if the right side of (15) were a fractional linear function. It will be clear from what follows that M is a center in this case also. We integrate (15):

$$\frac{y}{h}\left\{[h^2 - h(R+h_0) + h_1h_2]\left[\left(\frac{y}{h}\right)^{\frac{m+1}{m}} - 1\right] - \frac{m+1}{m}h_1h_2\right\} = C.$$

We examine the flow line for C = 0:

$$1.\ \ y = 0,\ h \leqslant H_1,$$

$$2.\quad \left(\frac{y}{h}\right)^{1+\frac{1}{m}} = \frac{h^2 - h(h_0+R) + \frac{2m+1}{m}h_1h_2}{(h-h_1)(h_2-h)}, \quad h \geqslant H_1. \tag{17}$$

The curve defined by (17) meets the vx axis at N_1 and N_2. The zero flow line is a loop with two branches that have an asymptote, whose equation is of the form $x = x_2$. It is readily shown that M lies within the region bounded by the zero flow line. The two different integral curves for the differential equation intersect only at the singular points of that equation if the functions in the numerator and denominator on the right in (15) are continuous, as are their first partial derivatives with respect to x and y. The closure of the zero flow line implies that M is a center (vortex point) and that all flow lines having $C > 0$ are closed lines.

The reserve of material, i.e., the volume not involved in the rolling, is defined by

$$q = 2L\int_{X_1}^{x_s} \frac{h^2 - h(R+h_0) + \frac{2m+1}{m}h_1h_2}{(h-h_1)(h_2-h)}\, h\,dx.$$

This shows that q and the flow-line pattern are not dependent on the angular velocity of the rolls. The equation for q can be given the following form in terms of dimensionless quantities:

$$q = 2L\bar{R}^2 \int_{\bar{X}_1}^{\xi_s} \Big\{(1+\eta_0-\sqrt{1-\xi^2})^2 - (1+\eta_0)(1+\eta_0-\sqrt{1-\xi^2})$$

$$+ \frac{2m+1}{m}k\eta_0\,[1-\eta_0(k-1)]\Big\}\{[1-\eta_0(k-1)-\sqrt{1-\xi^2}](\sqrt{1-\xi^2}-k\eta_0)\}^{-1}d\xi.$$

It is important to produce films and sheets having uniform thickness by calandering, and this we now consider. We write (10) in terms of dimensionless quantities:

$$\int_{\xi_1}^{\xi_s}\{[\eta_0(k-1)+\sqrt{1-\xi^2}-1]\,|\eta_0(k-1)+\sqrt{1-\xi^2}-1|^{m-1}$$

$$\times(\sqrt{1-\xi^2}-k\eta_0|\sqrt{1-\xi^2}-k\eta_0|^{m-1}\}\{(1+\eta_0-\sqrt{1-\xi^2})^{2m+1}\}^{-1}d\xi = 0.$$

This allows us to calculate the relative thickness k of the emerging sheet, which is dependent on: 1) the minimum relative gap η_0; 2) the roll insertion depth ξ_s; 3) power m. We assume that m and ξ_s, generally speaking, vary within certain limits, while η_0 is a constant. We term the process stable if we have not only $|m - m^0| < \varepsilon_1$ and $|\xi_s - \xi_s^0| < \varepsilon_2$ but also $|k - k^0| < \varepsilon_3$ for η_0 constant, where m^0, ξ_s^0, k^0 and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$,

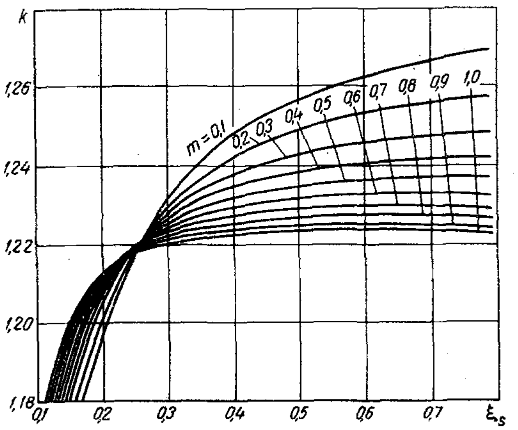

Fig. 3. Graph of $k(\xi_s)$ for various m with $\eta_0 = 0.0025$.

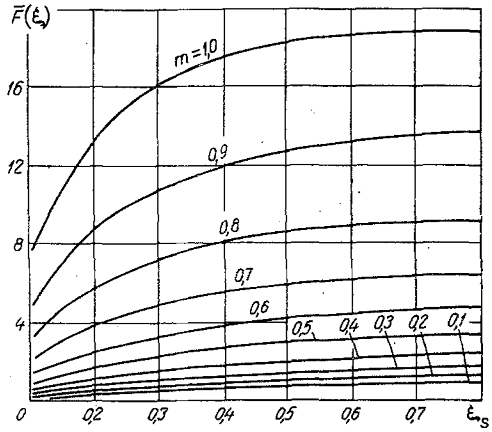

Fig. 4. Graphs of $\bar{F}(\xi)$ for various m with $\eta_0 = 0.0025$.

$\varepsilon_3 > 0$ are quantities specified in advance. The stability may be judged from the curves of Fig. 3. The process becomes more stable as m decreases and as ξ_S increases. Figure 4 shows that the frictional force varies little with ξ_S if m is small enough, but that $\bar{F}$ begins to increase rapidly if m approaches one.

A program has been written for the Minsk-2 computer in order to calculate the quantities.

NOTATION

τ	is the shear stress;
$\dot{\gamma}$	is the velocity gradient;
μ_0, m	are the rheological constants of the material;
u, v	are the velocity components along the Ox and Oy axes;
p	is the pressure;
x_S	is the loading thickness.

LITERATURE CITED

1. E. Bernhardt (editor), Processing of Thermoplastic Materials [Russian translation], Izd. Khimiya (1965).
2. N. G. Bekin and V. A. Nemytkov, Kauchuk i Rezina, No. 10 (1966).
3. S. M. Targ, Basic Topics in the Theory of Laminar Flow [in Russian], Gostekhizdat (1951).
4. N. A. Kozulin and I. A. Gorlovskii, Paint Industry Equipment [in Russian], Goskhimizdat (1959).
5. I. R. A. Pearson, J. Fluid. Mech., 7, 481 (1960).
6. R. V. Torner and G. V. Dobrolyubov, Kauchuk i Rezina, No. 4 (1958).
7. K. S. Malenko, Khim. Mash., No. 1 (1965).
8. R. Gaskell, J. Appl. Mech., 17, No. 2, 334 (1950).
9. J. M. McKelvey, Polymer Processing [Russian translation], Izd. Khimiya (1965).
10. R. Rautenbach, Chem. Ing. Tech., 34, No. 3, 231 (1962).
11. I. Tomita, Bull. JSME, 2, No. 7 (1950).
12. A. M. Evtel', Prikl. Matem. i Mekh., 3, No. 2 (1939).
13. N. V. Roze, V. N. Krasovskii, and R. G. Mirzoev, Polymer-Working Techniques and Machines [in Russian], Izd. Tekh. Inst. im. Lensoveta, Leningrad (1967).
14. G. Sansone, Ordinary Differential Equations [Russian translation], Vol. 2, IL (1954).